Formulae to 'Three Problems - One Solution'

Floating Bodies of Equilibrium in two dimensions, the Tire Track Problem and Electrons in a Parabolic Magnetic Field

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Here some formulae and results complete the explanations of http://www.tphys.uni-heidelberg.de/∼wegner/Fl2mvs/Movies.html.

More details are given in [1]. The first section gives some basic formulae, the second section contains formulae in the limit, in which the curves can be represented by 'elementary' functions like exponential- and trigonometric functions, whereas in the general case double-periodic functions are needed.

1 Few Basic Formulae

1.1 Hydrostatics of the Problem

I denote the mass of the body by m, its relative density, that is the ratio of the density of the body to the density of water, in which it floats, by ρ_d. The body displaces its mass in the fluid according to Archimedes. Then the part of the mass of the body floating below the waterline is $m_2 = m\rho_d$; the part of the mass floating above the waterline is $m_1 = m(1-\rho_d)$. Correspondingly the part of the cross-section below the waterline is given by $\mathcal{A}_2 = \rho_d\mathcal{A}$, the part above the waterline by $\mathcal{A}_1 = (1-\rho_d)\mathcal{A}$, where $\mathcal{A}$ is the total cross-section.

Denoting the height of the center of gravity of mass m_1 above the waterline by h_1 and the depth of the center of gravity of m_2 below the waterline by h_2, then the potential energy of the body is

$$\mathcal{V} = m_1 g h_1 + (m - m_2) g h_2 = m(1-\rho_d) g (h_1 + h_2), \tag{1}$$

since measured from the waterline the mass m_1 is lifted by h_1, and the mass m_2 is lowered by h_2, whereas the mass m of the water is lifted by h_2. The distance of both centers of gravity is given by $h = h_1 + h_2$. It follows that h is independent of the orientation, since the potential energy has to be equal in all orientations.

If one rotates the body by an infinitesimal angle $\delta\phi$ to the left, then the centers of gravity are shifted to the right by

$$\left(-h_1 + \frac{2\ell^3}{3\mathcal{A}_1}\right)\delta\phi, \text{ upper center of gravity} \tag{2}$$

$$\left(h_2 - \frac{2\ell^3}{3\mathcal{A}_2}\right)\delta\phi, \text{ lower center of gravity.} \tag{3}$$

2ℓ is the length of the waterline, that is the length of the line, that separates the part of the cross-section above the waterline from that below this line.

Both centers of gravity have to be vertically above each other, since otherwise a torque acts on the body, both displacements have to be equal. This yields

$$\frac{2}{3}\ell^3\left(\frac{1}{\mathcal{A}_1} + \frac{1}{\mathcal{A}_2}\right) = h_1 + h_2. \tag{4}$$

Thus ℓ has to be constant.

The infinitesimal rotation is performed around the midpoint of the waterline, since during the rotation the same area emerges out of the water on one side as is immersed on the other side. The midpoint of the waterline moves in direction of the waterline. These midpoints constitute the envelope of the waterlines (in the figures and animations depicted in red). Thus the requirement is: Find an envelope so that the tangents to this envelope at distance ℓ from the tangential point lie in both directions on one and the same curve.

1.2 The Differential Equation

The differential equation for the curve is derived in [2] by means of the following consideration: The radius r of the boundary as function of the polar angle ψ is expanded in powers of an expansion parameter ϵ (Taylor expansion) starting from a circle of radius r_0, where

$$r(\psi) = r_0\big(1 + \epsilon\cos(p\psi) + \sum_{n=2}^{\infty} c_n(\epsilon)\cos(np\psi)\big), \tag{5}$$

is set with $c_n(\epsilon) = O(\epsilon^n)$. The resulting eqs. yield in first order of ϵ a solution for $p-2$ different densities for given p. Continuation of the calculation by means of computer algebra yields surprisingly that all $p-2$ solutions are identical up to seventh order in ϵ, where I discontinued the calculation. The conjecture that this holds in higher orders suggested itself.

Naturally p has to be integer, so that the border yields a closed curve after one revolution. If instead one allows a 'border curve', whose distance from the origin oscillates periodically as function of the polar angle, but for a value of p, which differs infinitesimally from an integer, then the curve will deviate from the original one after a revolution around the origin by an infinitesimal angle χ. If we require that there is an envelope, whose tangent connects these both parts of the 'border' curve at distance ℓ, then it is possible to derive a differential equation for the curve. It is non-linear and of third order. ℓ^3 is proportional to χ in the limit of small χ. The ratio yields a constant of proportionality a,

$$a = \lim_{\chi\to 0}\frac{3\chi}{16\ell^3}. \tag{6}$$

The differential equation in [2], which I do not reproduce here, yields after two integrations

$$\frac{1}{\sqrt{r^2+(\frac{\mathrm{d}r}{\mathrm{d}\psi})^2}} = ar^2+b+cr^{-2} \tag{7}$$

with the integration constants b and c.

1.3 Extreme Radii

Maximal and minimal distance of the curve from the origin is obtained from $\frac{\mathrm{d}r}{\mathrm{d}\psi}=0$. Thus eq. (7) yields the equation

$$ar_i^4+br_i^2-r_i+c=0 \tag{8}$$

for the extreme radii. Given the maximal and minimal distance from the origin (center), one obtains two conditions on the three coefficients a, b und c. There remains one free condition. It can be used to fix, how many (p) oscillations between maximal and minimal distance close the curve.

Since eq. (8) is of fourth order, it may have up to four real solutions. How many are real, depends on the coefficients a, b und c. An extensive discussion can be found in section 2.1 of [1]. There can be no real solution or two or four. If there are two real solutions, then there is one curve. If there are four real solutions, then there are two curves. One sees from eq. (8) that the sum of the four solutions is zero. Actually some r_i can be negative. Generally the extreme radii are given by $|r_i|$. Since their sum vanishes, they can be written

$$r_{4,3}=r_0(1\pm\epsilon), \quad r_{2,1}=-r_0(1\pm\hat{\epsilon}). \tag{9}$$

$\hat{\epsilon}$ can be either real (two curves) or purely imaginary (one curve).

1.4 Differential equations once more

In order to solve the equations it is useful to introduce a parameter representation. One introduces the parameter u, which measures the distance along the perimeter like milestones along a road. For this u one has

$$\frac{\mathrm{d}u}{\mathrm{d}\psi} = \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\psi}\right)^2} \tag{10}$$

This allows to rewrite eq. (7)

$$\left(\frac{\mathrm{d}q}{\mathrm{d}u}\right)^2 = -4a^2\prod_{i=1}^{4}(q-q_i), \tag{11}$$

where q denotes the square of the radius, $q=r^2$, $q_i=r_i^2$. This eq. is solved by the integral

$$u=\int\frac{\mathrm{d}q}{2a\sqrt{-\prod_{i=1}^4(q-q_i)}}. \tag{12}$$

Then the equation for the angle ψ reads

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = aq + b + cq^{-1}, \tag{13}$$

which is solved by the integral

$$\psi(u) = \int \mathrm{d}u (aq + b + cq^{-1}). \tag{14}$$

1.5 The Solution

In general these integrals cannot be expressed by elementary functions. They are expressed by Weierstrass functions. They can be found in the tables of functions [3, 4, 5], but also in books on function theory like [6]. One obtains from eq. (12)

$$q(u) = q_i \frac{\wp(u) - \wp(3v)}{\wp(u) - \wp(v)}, \quad q_i = r_i^2 = \left(\frac{\wp(2v) - \wp(v)}{2a}\right)^2 \tag{15}$$

with the Weierstrass $\wp$ function (pronounced p function). This function is double periodic. One of the periodicities yields the periodic oscillation of the radius. v is given by

$$\wp(2v) = \frac{4ca - b^2}{3}. \tag{16}$$

Since the eq. determines $2v$, one can add to a solution v one of the three half-periods and obtains four different sets of $\wp(v), \wp(3v)$, which belong to the four different extreme radii r_i.

Moreover the Weierstrass functions depend on the two invariants g_2 and g_3. They are related to the constants a, b and c by

$$g_2 = \frac{4}{3}(4ca - b^2)^2 + 8ab, \tag{17}$$

$$g_3 = -\frac{8}{27}(4ca - b^2)^3 - \frac{8}{3}ab(4ca - b^2) + 4a^2. \tag{18}$$

Carrying out the integral (14) and combining x and y to a complex number, which in the complex plane represents the curve, one obtains

$$z(\chi, u) := x + \mathrm{i}y = \mathrm{e}^{\mathrm{i}\psi(u)} r(u) \tag{19}$$

with

$$z(\chi, u) = \frac{\mathrm{e}^{\mathrm{i}\chi + 2u\zeta(2v)}}{2a\sigma^2(2v)} \frac{\sigma(u - 3v)}{\sigma(u + v)}. \tag{20}$$

ζ is the negative of the integral of $\wp$, σ the exponential of the integral of ζ.

One can choose a purely imaginary v for all curves. The distance

$$2\ell = |z(v, \chi, u + \delta u) - z(\hat{v}, \hat{\chi}, u - \delta u)|, \tag{21}$$

is independent of u, provided

$$e^{i(\chi-\hat{\chi})} = \mp e^{-2\delta u(\zeta(2v)-\zeta(2\hat{v}))} \frac{\sigma(2\delta u + v + \hat{v})}{\sigma(2\delta u - v - \hat{v})} \tag{22}$$

holds. The minus sign applies, if one considers two equal curves rotated by the angle $\chi - \hat{\chi}$ against each other; the plus sign applies for two different curves. The length of the distance is obtained from

$$4\ell^2 = \frac{1}{\wp(2\delta u + v - \hat{v}) - \wp(2v)}. \tag{23}$$

2 The limit case yielding elementary functions

In the limit case, in which two extreme radii are equal, the Weierstrass functions degenerate to single-periodic functions. One of the solutions for the curve is a circle of radius r_0. The two other extreme radii are $r_0(1+\epsilon)$ and $r_0|\epsilon - 1|$.

2.1 The periodic case $\epsilon > 2$

In this case the two smallest radii $|r_i|$ are equal and one obtains

$$z(\chi, u) = r_0 \frac{(\epsilon^2-2)\cos(2\lambda u) + i\epsilon\sqrt{\epsilon^2-4}\sin(2\lambda u) - \epsilon}{\epsilon - 2\cos(2\lambda u)} e^{i(\chi - u/r_0)} \tag{24}$$

with

$$\lambda = \frac{\sqrt{\epsilon^2-4}}{2\epsilon r_0}. \tag{25}$$

Expressed by the Cartesean coordinates x and y eq. (24) yields

$$x(\chi, u) = c_1(u)\cos(\chi - u/r_0) - s_1(u)\sin(\chi - u/r_0), \tag{26}$$
$$y(\chi, u) = c_1(u)\sin(\chi - u/r_0) + s_1(u)\cos(\chi - u/r_0) \tag{27}$$

with

$$c_1(u) = r_0 \frac{(\epsilon^2-2)\cos(2\lambda u) - \epsilon}{\epsilon - 2\cos(2\lambda u)}, \tag{28}$$
$$s_1(u) = r_0 \frac{\epsilon\sqrt{\epsilon^2-4}\sin(2\lambda u)}{\epsilon - 2\cos(2\lambda u)}. \tag{29}$$

The distance 2ℓ between the curve points $\chi, u+\delta u$ and $\hat{\chi}, u-\delta u$ is given by

$$\begin{aligned} 4\ell^2 &= |z(\chi, u+\delta u) - z(\hat{\chi}, u-\delta u)|^2 \\ &= (x(\chi, u+\delta u) - x(\hat{\chi}, u-\delta u))^2 + (y(\chi, u+\delta u) - y(\hat{\chi}, u-\delta u))^2 \end{aligned} \tag{30}$$

If δu obeys

$$\tan(2\lambda\delta u) = 2\lambda r_0 \tan\left(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}\right), \tag{31}$$

then the distance is independent of u with

$$2\ell = \frac{2r_0}{\sqrt{1+\frac{\epsilon^2-4}{\epsilon^2}\cot^2(2\lambda\delta u)}}. \tag{32}$$

2.2 The case $\epsilon < 2$

In this case the two middle radii are equal. One obtains two curves, which approach asymptotically the circle. The curve with $s = +1$ lies outside the circle of radius r_0, the other one with $s = -1$ inside this circle,

$$z(\chi, u) = r_0 \frac{(2-\epsilon^2)\cosh(2\lambda u) + \mathrm{i}\epsilon\sqrt{4-\epsilon^2}\sinh(2\lambda u) + s\epsilon}{2\cosh(2\lambda u) - s\epsilon} \mathrm{e}^{\mathrm{i}(\chi - u/r_0)} \tag{33}$$

with[1]

$$\lambda = \frac{\sqrt{4-\epsilon^2}}{2\epsilon r_0}. \tag{34}$$

Expressed in Cartesean coordinates x and y one obtains from eq. (33)

$$\begin{aligned} x(\chi,u) &= c_2(u)\cos(\chi - u/r_0) - s_2(u)\sin(\chi - u/r_0), && (35)\\ y(\chi,u) &= c_2(u)\sin(\chi - u/r_0) + s_2(u)\cos(\chi - u/r_0) && (36)\end{aligned}$$

with

$$\begin{aligned} c_2(u) &= r_0 \frac{(2-\epsilon^2)\cosh(2\lambda u) + s\epsilon}{2\cosh(2\lambda u) - s\epsilon}, && (37)\\ s_2(u) &= r_0 \frac{\epsilon\sqrt{4-\epsilon^2}\sinh(2\lambda u)}{2\cosh(2\lambda u) - s\epsilon}. && (38)\end{aligned}$$

If δu satisfies

$$\tanh(2\lambda\delta u) = 2\lambda r_0 \tan\left(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}\right), \tag{39}$$

then the distance between the points $\chi, u + \delta u$ and $\hat{\chi}, u - \delta u$ on two curves, which are both outside or both inside the circle of radius r_0 is independent u

$$2\ell = \frac{2r_0\epsilon}{\sqrt{(4-\epsilon^2)\coth^2(2\lambda\delta u) + \epsilon^2}} \tag{40}$$

The distance between the point $\chi, u+\delta u$ on the curve outside the circle and the point $\hat{\chi}, u - \delta u$ inside the circle of radius r_0 is constant 2ℓ

$$2\ell = \frac{2r_0\epsilon}{\sqrt{(4-\epsilon^2)\tanh^2(2\lambda\delta u) + \epsilon^2}}, \tag{41}$$

if δu satisfies

$$\tanh(2\lambda\delta u) = 2\lambda r_0 \cot\left(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}\right). \tag{42}$$

[1] The hyperbolic functions are defined by $\cosh(z) = (\mathrm{e}^z + \mathrm{e}^{-z})/2$, $\sinh(z) = (\mathrm{e}^z - \mathrm{e}^{-z})/2$, $\tanh(z) = \sinh(z)/\cosh(z)$, $\coth(z) = \cosh(z)/\sinh(z)$.

2.3 The limit case $\epsilon = 2$

In the limit case $\epsilon = 2$ the curve is described by

$$z(\chi, u) = \frac{3r_0 + \mathrm{i}u}{1 - \mathrm{i}u/r_0} \mathrm{e}^{\mathrm{i}(\chi - u/r_0)} \tag{43}$$

or equivalently by

$$x(\chi, u) = c_3(u)\cos(\chi - u/r_0) - s_3(u)\sin(\chi - u/r_0), \tag{44}$$
$$y(\chi, u) = c_3(u)\sin(\chi - u/r_0) + s_3(u)\cos(\chi - u/r_0) \tag{45}$$

with

$$c_3(u) = \frac{r_0(3r_0^2 - u^2)}{r_0^2 + u^2}, \tag{46}$$
$$s_3(u) = \frac{4ur_0^2}{r_0^2 + u^2}. \tag{47}$$

If

$$\frac{\delta u}{r_0} = \tan(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}), \tag{48}$$

then the distance between the two points on the curves $\chi, u + \delta u$ and $\hat{\chi}, u - \delta u$ is constant 2ℓ with

$$\ell^2 = \frac{r_0^2 \delta u^2}{r_0^2 + \delta u^2}. \tag{49}$$

2.4 Distance to the circle

The circle with radius r_0,

$$\hat{z}(u) = r_0 \mathrm{e}^{-\mathrm{i}u/r_0} \tag{50}$$

is a second solution for $\epsilon > 2$. For $\epsilon \leq 2$ the curves approach asymptotically the circle for $u \to \infty$. In all these cases the distance between the curve at $\chi = 0, u$ and the circle at u is constant $2\ell = \epsilon r_0$.

2.5 Linear case

Finally there is a solution, where the curve approaches asymptotically a straight line. With $s = \pm 1$ both curves are described by

$$x(u) = \frac{sd}{\cosh(2u/d)}, \tag{51}$$
$$y(y_0, u) = y_0 - u + d\tanh(2u/d). \tag{52}$$

Then the points $y_0, u + \delta u$ and $\hat{y}_0, u - \delta u$ on two curves have the distance

$$2\ell = |y_0 - \hat{y}_0 - 2\delta u|, \tag{53}$$

if

$$y_0 - \hat{y}_0 - 2\delta u = \begin{cases} -d\tanh(2\delta u/d) & \text{same } s \\ -d\coth(2\delta u/d) & \text{different } s. \end{cases} \tag{54}$$

References

[1] F. Wegner, *Floating Bodies of Equilibrium in 2D, the Tire Track Problem and Electrons in a Parabolic Magnetic Field* http://xxx.uni-augsburg.de/abs/physics/0701241

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