Floating Bodies of Equilibrium
A short survey

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Stanislaw Ulam asks in the Scottish Book [1] (problem 19), whether a sphere is the only solid of uniform density which will float in water in any position. In [3] I considered the two-dimensional problem for $\rho \neq 1/2$. (The case $\rho = 1/2$ has been solved by Auerbach [2]). I obtained a perturbation expansion around the circular case with a p-fold rotation axis. Within this expansion it turned out, that one and the same boundary is solution for $p-2$ different densities.

Recently [4] I was able to obtain a differential equation for the boundary

$$\frac{1}{\sqrt{r^2 + r'^2}} = ar^2 + b + cr^{-2}, \tag{1}$$

with $r' = \mathrm{d}r/\mathrm{d}\psi$. This equation can be integrated

$$\begin{aligned} r'^2 &= \frac{r^4}{(ar^4 + br^2 + c)^2} - r^2, & (2) \\ \psi &= \pm \int \frac{\mathrm{d}r(ar^4 + br^2 + c)}{r\sqrt{r^2 - (ar^4 + br^2 + c)^2}} \\ &= \pm \int^{r^2} \frac{\mathrm{d}q(aq^2 + bq + c)}{2q\sqrt{q - (aq^2 + bq + c)^2}} & (3) \end{aligned}$$

and is thus given by an elliptic integral. With increasing ψ the radius r oscillates periodically between the largest and the smallest radii $r_>$ and $r_<$, resp. The three constants a, b, and c are determined by these extreme radii, and by the periodicity of the boundary. Since r' vanishes for the extrema of r, one has

$$r_{>,<} = ar_{>,<}^4 + br_{>,<}^2 + c. \tag{4}$$

The periodicity is given by

$$\int_{r_<}^{r_>} \frac{\mathrm{d}r(ar^4 + br^2 + c)}{r\sqrt{r^2 - (ar^4 + br^2 + c)^2}} = \frac{\pi}{p}. \tag{5}$$

In the near-circular case $(r_> - r_<)/(r_> + r_<) \ll 1$ one has $p^2 \approx 1 + 8ar^2$. The ratio $(r_> - r_<)/(r_> + r_<)$ has to be so small, that the water-line cuts the boundary in only two points.

The boundary curves have the following interesting property: Suppose we know two points L and R on the curve (with $r_L \neq r_R$), so that the angles β_L and β_R between the chord LR and the tangents on the curves are equal. Let us then move the same piece of perimeter from both points along the curve to L' and R', then also $\beta_{L'} = \beta_{R'}$, and the distance $L'R'$=LR is unchanged. This is exactly the property that has to be obeyed by the water-line [3]. It has to be of constant length $2l$ at all orientations and it is obvious, that the area above and below the water-line stays constant.

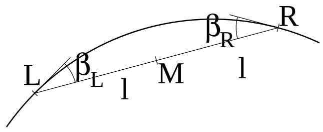

Figure 1: Boundary curve, water-line LR, tangents, and the angles β.

We now fix the orientation of the boundary and rotate the gravitational field and the water-surface. Then the midpoints M of the chords LR generate the evolute of the water-lines; that is the water-lines are tangent to this evolute.

I show an example for $p = 7$.

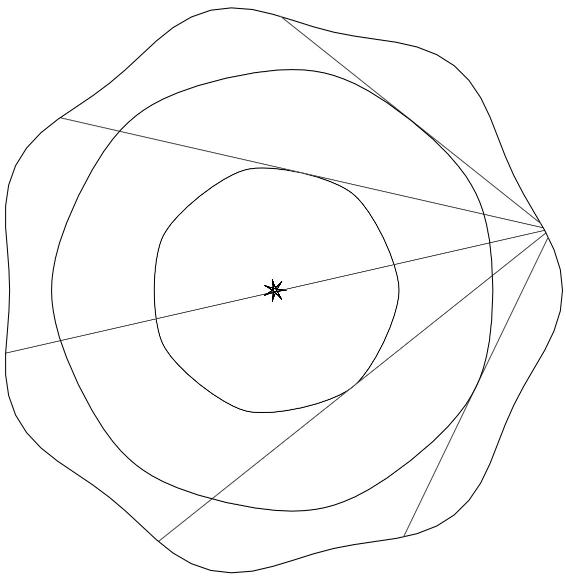

Figure 2: Boundary for $p = 7$. The outermost curve is the boundary, the inner curves are the envelopes of the water-lines. The innermost one for density $\rho = 1/2$, the two others for densities 0.22748 and 0.77252, and for 0.05269 and 0.94731. Some water-lines are shown.

How did I find the differential equation? First of all I conjectured from the perturbation expansion in [3], that one and the same boundary can float at different densities in arbitrary orientation. Next I came to the conjecture, that the condition, that the periodicity p is an integer, is a physical requirement, but probably not a mathematical requirement for obtaining curves with the above mentioned property. If p differs infinitesimally from an integer, then going once around the angle 2π one arrives at the same function shifted, however, in ψ-direction by an infinitesimal angle 2α. I conjectured that there is also a line between L and R belonging to angles $\psi_{L,R}$ differing by $\delta\psi \approx 2\pi$ of constant small length $2l$, which obeys the condition for the angles $\beta_L = \beta_R$. Equivalent is the condition that one can obtain L and R by going from an evolute tangentially a distance l in both directions. In the limit of infinitesimal l this leads to a differential equation for the curve, which can be integrated to eq. (1), [4].

The above-mentioned property holds also between two different curves with identical constants a, b, and c, thus between two curves rotated by an arbitrary angle against each other.

[1] R.D. Mauldin (ed.), *The Scottish Book*, Birkhäuser Boston 1981
[2] H. Auerbach, *Sur un probleme de M. Ulam concernant l'equilibre des corps flottant*, Studia Math. 7 (1938) 121-142
[3] F. Wegner, *Floating Bodies of Equilibrium*, e-Print archive physics/0203061
[4] F. Wegner, *Floating Bodies of Equilibrium II*, e-Print archive physics/0205059